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Applying Bayesian updating to CPT data analysis

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ABSTRACT: Evaluation of geotechnical parameters on a project site is a necessary step in geotechnical engineering. However, due to the inherent variability of soil properties and the lack of data, many unavoidable uncertainties arise during a site-specific geotechnical characterization. This challenging task can be addressed under the Bayesian framework. The aim of this paper is to apply the Bayesian approach to a reference example of friction angle evaluation in sand, using the Bayesian Equivalent Sample Toolkit (BEST). BEST is an Excel VBA program for probabilistic characterization of geotechnical properties. In particular, in this study the statistical analysis has been performed using CPT tests from reference field studies. The results obtained for one case study involving CPT are discussed.

1 INTRODUCTION

A major application of CPTu is in the evaluation of material properties for geotechnical units identified within a site. The description of a given unit property (e.g. the friction angle of a sand, ϕ') may be done in deterministic terms (a single value, eventually varying with depth) or in probabilistic terms. The latter approach is increasingly important, as the treatment of material uncertainties in geotechnical design becomes more formalized (e.g. in the selection of characteristic values for partial factor limit state design or in the application of direct reliability evaluation methods). The variability of geological processes is one fundamental component of a variation of geotechnical properties. Such variability is known in literature as inherent variability or "aleatory uncertainty". This irreducible component is increased by "epistemic uncertainty", comprising factors such as statistical error (lack of data), measurement error (inadequate equipment and/or operator errors), and transformation uncertainties (Phoon & Kulhawy 1999a). The latter component is particularly important for CPTu data, which may be directly used in design, but most frequently requires transformation models to obtain derived values. Within this context, Bayesian updating is an emerging framework (Gelman et al. 2013) well suited to handle the problem of material property evaluation under uncertainty conditions.

Within geotechnics, Wang & Cao (2013) developed an equivalent sample method for Bayesian analysis in which different components of epistemic uncertainty may be treated. To do so the limited data acquired during site-specific investigation is combined with the so called prior knowledge, representing the information about the geotechnical property before any observation data are collected. The integrated knowledge (posterior knowledge), is then used to obtain a precise statistical description of the property by means of a numerical procedure known as Markov Chain Monte Carlo simulation. In order to avoid mathematical and statistical hurdles, Wang et al. (2016) developed an Excel toolkit, called Bayesian Equivalent Samples Toolkit (BEST), to evaluate the probability distribution and nominal value of the design parameters. The aim of this paper is to perform a statistical analysis, through BEST, of one design example, in which the observation data consist of four CPT tests.

2 UNCERTAINTY MODELLING

2.1 *Inherent variability*

Consider for instance a sand deposit, and its friction angle ϕ' as the design parameter X_D. According to Wang et al. (2015), which reports the probability distribution of different geotechnical parameters, inherent variability of friction angle can be modelled using a normal random variable with a mean μ and standard deviation σ:

$$\phi' = \mu + \sigma \cdot z; \tag{1}$$

where z is a standard gaussian random variable. Note that the spatial variability of the friction angle is not considered herein.

2.2 *Transformation model*

In this study a linear semi-log regression, developed by Kulhawy & Mayne (1990), between ϕ′ and the normalized cone tip resistance q is considered (Fig. 1).

The regression equation is defined as:

$$\xi = \ln q = a\phi' + b + \varepsilon; \tag{2}$$

where q is the normalized cone tip resistance $q = (q_c/p_a)/(\sigma'_{v0}/p_a)^{0.5}$, which is a function of the effective stress σ'_{v0} and the atmospheric pressure p_a. The use of a stress-normalized regression like this implies a certain assumption about the effect of stress increases with depth on the data. The regression coefficients have values a = 0.209, b = –3.684, while the transformation error, ε, is a Gaussian random variable defined by a zero mean and a standard deviation $\sigma_m = 0.586$. It is worth noting that Equation 2 considers the transformation uncertainty of the regression model, through the ε term. Replacing Equation 1 in Equation 2 leads to:

$$\xi = \ln q = (a\mu + b) + a\sigma z + \sigma_m \varepsilon; \tag{3}$$

Thus, the normalized observation data ξ are treated as a Gaussian random variable with mean

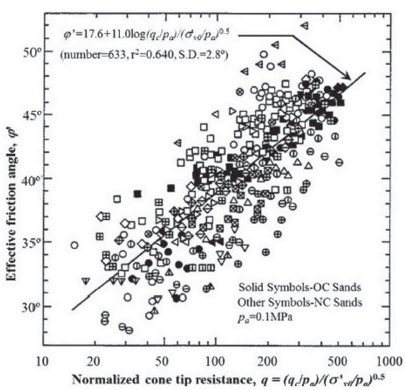

Figure 1. Regression analysis between ϕ′ and normalized cone tip resistance q (Kulhawy & Mayne 1990).

(aμ + b) and standard deviation $\sqrt{((a\sigma)^2 + \sigma_m^2)}$. The expression obtained makes clear that inherent variability (z) is independent from transformation uncertainty (ε).

3 BAYESIAN FRAMEWORK

A key feature of Bayesian statistical approach is the integration of the prior knowledge of the model parameters (i.e. μ, σ of ϕ′) with the observation data (i.e. CPT), to obtain the posterior knowledge of the model parameters. Indeed, the latter can be expressed through Bayes theorem as (Cao & Wang 2013):

$$P(\mu, \sigma \mid Data, prior) = KP(Data \mid \mu, \sigma)P(\mu, \sigma); \tag{4}$$

where:

- Data = ξ_i with i = 1, 2 … n_c, are the observation data obtained during the site investigation in the sand layer;
- P (μ, σ) reflects the prior distribution of the model parameters μ and σ of X_D (ϕ′), without considering the observation data;
- P (Data| μ,σ) is the so called likelihood function, which expressed the probability density function (PDF) of the observation data, for a given set of μ and σ;
- $K = [\int_{\mu,\sigma} P(Data|\ \mu,\sigma)\ P(\mu,\sigma)d\mu d\sigma]^{-1}$ represents the normalizing constant which does not depend on μ and σ;
- P (μ,σ| Data,prior) is the update or posterior knowledge of the model parameters μ and σ of the design parameter X_D (ϕ′).

Note that from now on P (μ,σ| Data,prior) will be denoted as P (μ,σ| Data).

3.1 *Prior knowledge*

Two different types of prior distribution of the model parameters can be considered in the Bayesian approach:

- non informative (i.e. uniform distribution);
- informative (i.e. histogram, triangular, conjugated distribution).

The former is considered when no prior information on X_D is available for the project site. In this case a rough estimation of the prior knowledge was obtained through a literature review (Phoon & Khulawy 1999a, Cabalar 2010, Salgado et al. 2000, Naeini & Baziar 2004). Only the range of model parameters has to be considered; indeed, the prior is defined as a joint uniform distribution as:

$$P(\mu,\sigma)=\begin{cases}\dfrac{1}{\mu_{\max}-\mu_{\min}}\times\dfrac{1}{\sigma_{\max}-\sigma_{\min}}; & \text{for } \mu\in[\mu_{\min},\mu_{\max}] \text{ and } \sigma\in[\sigma_{\min},\sigma_{\max}]\\ 0; & \text{otherwise}\end{cases} \tag{5}$$

Informative priors can be adopted when more information on X_D are available. As explained in the next section the use of such priors may increase the computational demands. Within the context of geotechnical engineering, a detailed analysis of available procedures for prior knowledge estimation is given in Cao et al. (2016).

3.2 *Likelihood function*

The likelihood function is defined as the PDF of the observation data for a given set of μ and σ. The Hypothesis of observation data as n_c independent Gaussian random variable, leads to:

$$P(Data\,|\,\mu,\sigma)=\prod_{i=1}^{nc}\frac{1}{\sqrt{2\pi}\sqrt{(a\sigma)^2+(\sigma_m)^2}}\times \mathrm{e}^{\left(-0.5\left(\frac{\zeta_i-(a\mu+b)}{\sqrt{(a\sigma)^2+(\sigma_m)^2}}\right)^2\right)} \tag{6}$$

It is worth noting that, when X_D (i.e. φ′) is not measured directly, the likelihood is a function of both the probabilistic model M_P, used to model the inherent variability, and the transformation model M_T (Eq. 2). Hence, given a set of observation data, a comparison among different M_T can be carried out by evaluating Equation 6 for each transformation model. In this way the most suitable likelihood function for a better estimation of the posterior knowledge can be selected (Cao & Wang 2014).

3.3 *PDF of the friction angle*

Once the posterior knowledge is evaluated, the probability density function of friction angle is then obtained through the theorem of total probability as:

$$P(\phi\,|\,Data,prior)=\int_{\mu,\sigma}P(\phi\,|\,\mu,\sigma)P(\mu,\sigma\,|\,Data)d\mu d\sigma; \tag{7}$$

where P(φ|μ, σ) is the probability density function of the friction angle for a given set of model parameters; P(μ,σ|Data) is the posterior knowledge of model parameters. This process can be repeatedly applied. If several data sets are available (i.e. CPT, SPT), the posterior knowledge of model parameters, after CPT results are considered, may be used as prior knowledge to obtain the posterior probability distribution using the SPT values as input data. Such an approach is called Bayesian Sequential Updating (BSU) (Cao et al. 2016).

3.4 *Markov Chain Monte Carlo (MCMC) simulation*

When the prior knowledge and likelihood function are sophisticated, Equation 7 can be hard to solve explicitly or analytically due to the presence of the normalizing constant K. To bypass this inconvenience, the MCMC method is used, which, in this case, draws many equivalent sample of φ′ from a target distribution, expressed by Equation 7. The samples drawn are then used for a statistical analysis of the parameter φ′. In particular, the Metropolis Hasting algorithm is used in the MCMC simulation (Hasting 1970, Wang & Cao 2013).

4 BAYESIAN EQUIVALENT SAMPLES TOOLKIT (BEST)

BEST is an Excel add-in that, using Excel VBA, implements MCMC to obtain Bayesian updates of geotechnical data statistics (Wang et al. 2016). The program is able to apply the Bayesian equivalent sample method for different soil types (i.e. clay, sand, and rock) with two different kinds of transformation models:

– built—in model;
– user—define model.

For the case of the "built-in model" option the transformation model M_T and its coefficients are already assigned. The inbuilt transformation models cover only a specific set of transformations, therefore requiring a particular set of input data. For the case of sand layers and friction angle the model 2.2 is featured, and the input data required are CPT data, and the prior knowledge of friction angle's model parameters (i.e. uniform distribution in the intervals [μ_{max}, μ_{min}], [σ_{max}, σ_{min}]). The "user-defined model" option is more flexible since is able to perform a statistical analysis of an arbitrary geotechnical parameter as long as the coefficients and uncertainty of M_T are specified. The program also requires the input of several numerical parameters that control the MCMC algorithm operation. These include the number of Markov iterations (i.e. number of MonteCarlo runs), the number of equivalent samples obtained in each MonteCarlo run, the number of samples to discard from one step to the next of the Markov iteration. If the proposal distribution used in the Metropolis Hasting algorithm is characterized by a wrong starting point, a large sequence of equivalent samples is necessary to reach the stationarity distribution of MCMC simulation, which is called the burn-in period (Ravenzwaaij et al. 2016). Hence, the number of samples to discard should be large, since the firsts will not be reliable for the subsequent statistical analysis. BEST suggests picking up more than 31000 equivalent samples and discarding more than 1000 of them.

4.1 *Illustrative example*

The following design example is taken from Brito & Sorensen (2010). This section shows how to evaluate the probabilistic distribution of friction angle for a dense fine glacial outwash sand deposit for the construction of a pad foundation. Four CPT tests were carried out to a depth up to 8 m, whose profiles are reported in Figure 2. Note that each CPT test consist in eighty data values.

Firstly, a Bayesian equivalent samples method is carried out using the built-in model for sand. Due to the lack of information about the project site, prior knowledge is described by a joint uniform distribution of the model parameters (μ, σ) of ϕ (Eq. 5). Initially, the ranges of prior model parameters are taken from BEST under the command help (Table 1). This contains the typical range values of μ and σ for different design parameters according to the literature.

The "built-in model" for the estimation of ϕ requires the normalized cone tip resistance as input data, and these are related to the friction angle through Equation 2. Initially, one single run is considered, in which 31000 equivalent samples are selected and 1000 of them are discarded. Figure 3a and 3b show, respectively, the histogram of the friction angle values drawn, with all CPT values as input data, and the scatter plot of the equivalent sample. The 90% confidence interval is reported since BEST evaluate the 5% percentile and the 95% percentile.

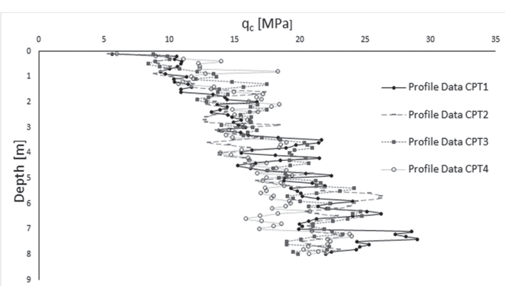

Figure 2. CPT data profiles.

Table 1. Range of prior values of μ and σ.

	μ_{max} (°)	μ_{min} (°)	σ_{max} (°)	σ_{min} (°)
Prior (BEST)	42	30	7.1	0.6
Prior (Phoon et al, 1996b)	41	35	6.97	1.27

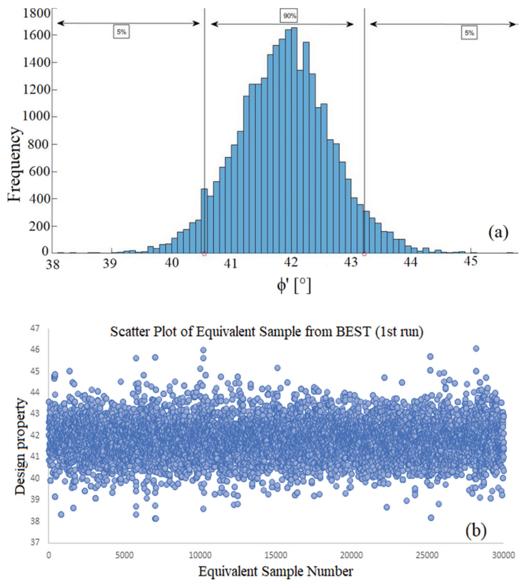

Figure 3. (a) Frequency of the friction angle drawn; (b) Scatter plot of the equivalent samples.

It is possible now to compare the obtained equivalent sample distribution to that of the original CPT data. To do so the equivalent samples generated are back-transformed into values of the measured property ξ through Equation 2 using the same coefficients a, b and σ_m adopted in the transformation model in BEST. In this way it is possible to evaluate 30000 values of the cone tip resistance q_c for a given depth D (assuming a constant bulk weight of 20 kN/m^3). The mean, the 5% percentile and the 95% percentile are plotted at each depth D alongside the measured data in Figure 4. The results show that almost all the data below 1 m are included in the 90% confidence interval. The anomaly at shallow depth points to some limitation of the transformation model M_T. Several reasons may be behind that anomaly: strength curvature at

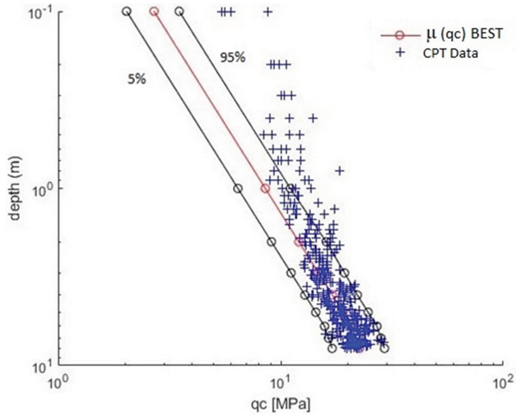

Figure 4. μ, 5% percentile, 95% percentile and data set.

very shallow stress levels, a relative increase in suction-derived effects and boundary effects on CPT tip resistance at shallow depths (Senders 2010).

4.2 *Effect of number of CPT input data*

The comparisons among the PDF and CDF of ϕ' for different single CPT_i tests, with i = 1, 2 … 4, and the one with all the observation data, as input data, are plotted in Figure 5a and 5b respectively. The results show that, increasing the observation data the standard deviation is significantly reduced (between 12 and 18%) with respect to the estimate obtained using a single CPT. On the other hand, the estimated value of μ is approximately constant. It seems reasonable that increasing the number of observation data, if all of them are equally reliable, there is a reduction of the estimated parameter uncertainties; indeed, the likelihood function become more relevant than the prior increasing the number of data sets.

4.3 *Effect of prior knowledge*

The influence of the prior knowledge is analyzed using a different range of the model parameter μ and σ, taken from Phoon & Kulhawy (1999b) and reported in Table 1. Note that, for this comparison, all the CPT data are used as input data. The results in terms of PDF(ϕ') and the statistics value for the two different cases are illustrated in Figure 6 and Table 3. It can be noticed that the difference between the two cases, in terms of 5% percentile, is 2.86°. Indeed, the prior distribution in BEST integrates the studies of Phoon & Kulhawy (1999b) with the ones of Cabalar 2010, Salgado et al. 2000, Naeini & Baziar 2004, in which micaceous sand, silty sand, and mixed and layered samples of sand are respectively analyzed. This provides a reduction of the prior range of the standard deviation

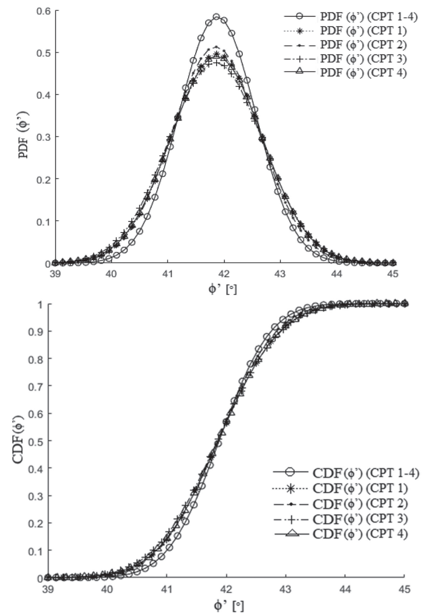

Figure 5. (a) PDF(ϕ') for different input data; (b) CDF(ϕ') for different input data.

Table 3. Range prior values from different sources.

Statistics	*No Prior*	*Prior* BEST	*Prior* Phoon & Kulhawy (1999b.)
μ (°)	43.3	41.87	40.93
σ (°)	1.13	0.68	1.83
5% (°)	42.02	40.73	37.87
95% (°)	45.9	43.92	43.96

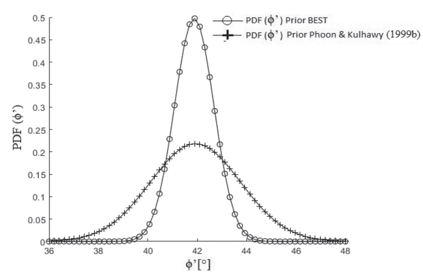

Figure 6. Probability density function of the friction angle using *prior* BEST and *prior* Phoon & Kulhawy (1999b).

and consequently a narrower estimation of PDF (ϕ'). Table 3 also includes the values that would be inferred from the data in the absence of any Bayesian updating: the friction estimate increases, skewed by the shallow data points.

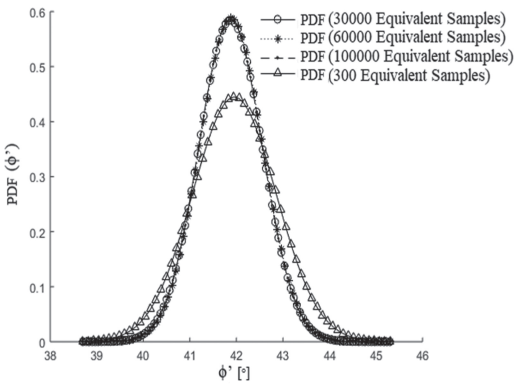

Figure 7. Probability density function of the friction angle for different sequences of equivalent samples.

4.4 *Effect of numerical parameters*

As mentioned in the previous section, a key step for the convergence of MCMC simulation is the definition of the number of equivalent samples to draw and the ones to discard for each run. The initial proposal distribution implemented in BEST, is characterized by a μ and σ equal to half of the prior mean and the standard deviation range (i.e. 36°, 3.25°), which represent a reliable starting point (Ravenzwaaij et al. 2016). To test this, several runs, with 310, 31000, 62000 and 110000 equivalent sample are conducted. The samples discarded for each run are respectively 10, 1000, 2000 and 10000. The results are illustrated in Figure 7. As suggested by BEST, 31000 equivalent samples are enough to reach convergence; indeed, no relevant difference exists among PDFs with more samples. On the other hand, as expected, a limited number of samples (i.e. 300) results in less accurate estimations, with broader probability distributions of ϕ′ and a wider 90% confidence interval.

5 CONCLUSION

In this paper, the Bayesian Equivalent Samples Toolkit, has been tested on a four CPT data set to estimate friction angle. As shown by the example analyses presented, this kind of numerical approach practically eliminates statistical error and concentrates the effort of the designer on the description of previous knowledge (prior) and in the selection of transformation models.

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